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Discrete Dynamical Systems with *SageMath*

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Abstract

In this study, we have developed interactive tools in SageMath *for the mathematical and geometric analysis of Discrete Dynamical Systems (DDS). By using these interactive tools, user can obtain the stability of fixed points, cobweb diagram, time series, and bifurcation diagram for one dimensional maps; phase diagram, time series, and bifurcation diagram for two-dimensional maps. In addition, we introduce a novel tool which helps us know what type of bifurcations (if any) occur for an entered two-dimensional DDS.*

1 Introduction

In order to investigate a dynamical system, beside the theory, one usually needs a computer algebra system (CAS) to make numerical and symbolic calculations and to obtain a visual representation of the system.

Currently, among a variety of computational software/tools/packages available on the field, some of them are more commonly used in the dynamical systems community. Mathematica (Dynamica for the book [10]), Maple (worksheets for the book [13]), and Matlab (programs and Simulink models for the book [12]) are the most popular commercial tools for the area of dynamical systems. There are also some popular non-commercial tools: For example, PyDSTool is a free simulation and analysis software for models of physical systems. It is written primarily in Python. AUTO 2000, which is a publicly available software for ordinary differential equations, was originally written in 1980 and is one of the most common software in the dynamical systems community. Another popular software is CONTENT. It is a multi-platform interactive environment to study dynamical systems. The current version supports bifurcation analysis of ODEs, iterated maps, and evolution PDEs in the unit interval. A freeware Dynamics Solver is intended to solve initial and boundary-value problems for continuous and discrete dynamical systems. It is possible to draw phase-space portraits, Poincaré maps, Liapunov exponents, cobweb diagrams, histograms, and bifurcation diagrams. Phaser provides the graphical and numerical simulations of differential and difference equations. While some of these tools are devoted solely on continuous dynamical systems, some investigate both discrete and continuous systems. There are also couple of articles and books dealing with the analysis of DDS with some popular computer algebra systems: [10, 18, 19] (DDS only), [13, 12, 14, 2] (both discrete and continuous systems).

According to the website of SageMath [20], the mission is described as follows: "Creating a viable free open source alternative to Magma, Maple, Mathematica and Matlab". With the motto "building the car instead of reinventing the wheel", SageMath brings together about 100 open source software packages and libraries that aims to address all computational areas of pure and applied mathematics [17, 7]. The open software SageMath was created by William Stein in 2005. For recent years, more and more researchers have been using it. SageMath is actively used in research mathematics and it is particularly strong in number theory, algebraic combinatorics, and graph theory [7]. There are also studies on mobile applications of SageMath [9]. For further examples, see 345 articles, 38 thesis, 37 books, and 55 preprints [20].

In the area of Dynamical Systems, there are some studies done with SageMath: In [8], the author presents an algorithm to determine all the rational preperiodic points for a morphism f defined over a given number field K, which is actually a part of an ongoing project on arithmetic (Number Theoretic) and complex dynamics with SageMath. The current state of the project can be found at `http://wiki.sagemath.org/dynamics/ArithmeticAndComplex`.

The computer algebra system Maxima, which is also one of the open-source packages of SageMath, is a descendant of Macsyma developed in the late 1960s at the Massachusetts Institute of Technology. Many later systems, such as Maple and Mathematica, were inspired by it. In [14], authors present an introduction to the study of chaos in discrete and continuous dynamical systems with *Maxima*. Their study covers many topics such as discrete and continuous logistic equation to model growth population, staircase plots (cobweb diagram), bifurcation diagrams and chaos transition, nonlinear continuous dynamics (Lorentz system and Duffing oscillator), Lyapunov exponents, Poincaré sections, fractal dimension, and strange attractors. Although, the study is comprehensive and covers many topics on one dimensional DDS in a clear way and gives the procedures in details, it doesn't present the two-dimensional discrete dynamical systems.

There are several advantages of SageMath: First of all, it is an open and free software [17]. SageMath, which was programmed in Python, has a client-server model (see Figure 1) which is well-adapted to the internet. Therefore, the output of the software can be seen in any computer connected to the internet. Unlike most mathematical tools, which require installation, SageMath does not need to be installed. It can be used in any location via the internet. SageMathCell project (`sagecell.sagemath.org`) is an easy-to-use web interface to SageMath. It allows us to create web pages with embedded interactive applets that can be used in a web browser. SageMathCell is built on top of the IPython architecture for executing Python code remotely.

The well-known computer algebra systems are designed mostly for professional researchers and require some mastery of programming syntax. This is an obstacle for inexperienced learners. In traditional computer algebra systems, an easy-to-use GUI (Graphic User Interface) is lacking. One of the

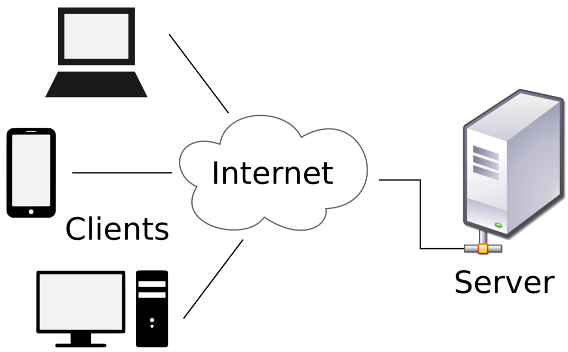


Figure 1: Client-server Model

main advantages of the presented tools is that they can be embedded in any web page and allow the user to enter input and see the output without programming.

The aim of this paper is to introduce a collection of interactive tools created by Sagemath in order to investigate one- and two-dimensional DDS. We develop mathematical and geometric tools for stability, cobweb diagrams, time-series, and bifurcation diagrams for one-dimensional DDS. For two-dimensional maps, we present tools for phase planes, time series, and bifurcation diagrams, and we propose a new tool with parametric parameter curve in tr-det plane. This tool helps us to know what types of bifurcations (if any) occur for an entered two-dimensional DDS.

This paper is organised as follows: In Section 2, we introduce the interactive tools for stability analysis of DDS. Detailed explanations of the tools will be given. Section 3 is devoted to tools for bifurcation analysis for one- and two-dimensional discrete systems. Basic definitions and theorems on stability and bifurcation can be found in Appendix A.

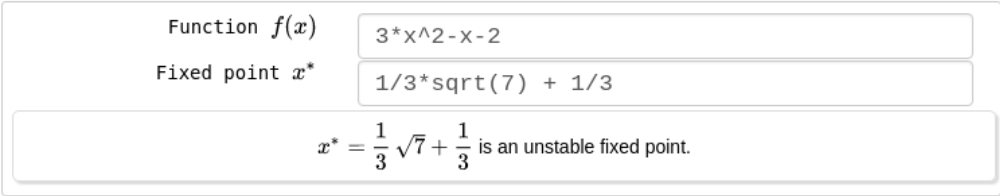
2 Stability with *SageMath*

In this section, we focus on the stability of fixed points of DDS. First, algebraic and geometric tools for one-dimensional maps are presented. Then we introduce the interactive tools for the stability of two-dimensional maps.

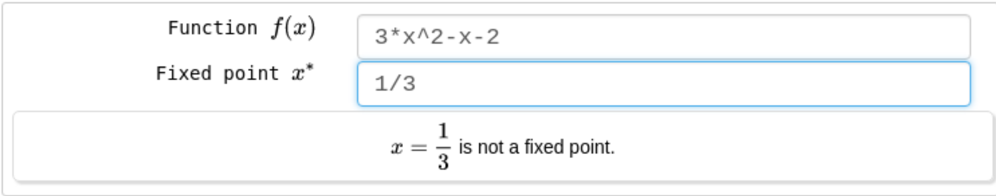
2.1 One-Dimensional DDS

The first tool we present determines the stability of a fixed point of a difference equation. The user inputs a difference equation $x_{n+1} = f(x_n)$ by entering a function f and one of the fixed points of f. Then the tool gives the stability information.

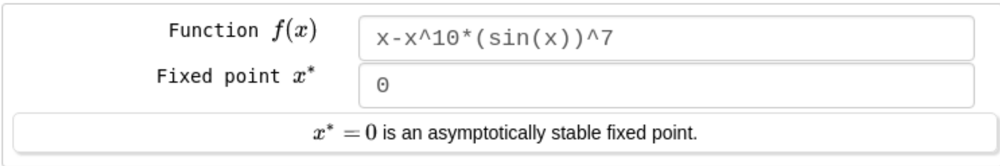
Finding the fixed points of a map is too difficult or even impossible in some cases, for example, when the map has a non-algebraic isocline equation. That is why the tool doesn't find any fixed points but the user should enter them.



(a)



(b)



(c)

Figure 2: Stability for one-dimensional maps [21]

The stability tool is particularly helpful for non-hyperbolic fixed points when the derivatives have to be evaluated many times. Figure 2 shows some examples of the use of the tool to examine the stability of one-dimensional maps. In Figure 2a, we find that the equilibrium point $x^* = \frac{1}{3}(1+\sqrt{7})$ of the one-dimensional map $x_{n+1} = 3x_n^2 - x_n - 2$ is unstable. When the entered point is not a fixed point for the given map, the tool gives that information as well, as shown in Figure 2b. Regarding Figure 2c, note that in order to determine the stability of the equilibrium point $x^* = 0$ of the map $x_{n+1} = x_n - x_n^{10}\sin^7 x_n$, we have to take derivatives seventeen times. For this map, the tool concludes that fixed point is asymptotically stable. Details about stability of one-dimensional maps can be found in Appendix A.

Now we introduce two graphical tools to determine the stability of fixed points for one-dimensional maps.

2.1.1 Cobweb and Time Series Diagrams

A *cobweb*, *staircase*, or *Verhulst* diagram is a visual method used in the study of discrete dynamical systems to investigate the qualitative behaviour of one-dimensional maps under iteration. Using a cobweb diagram, it is possible to analyze the long term evolution of an initial condition under repeated application of a map. More information on the use of cobweb diagrams can be found in [4, 5].

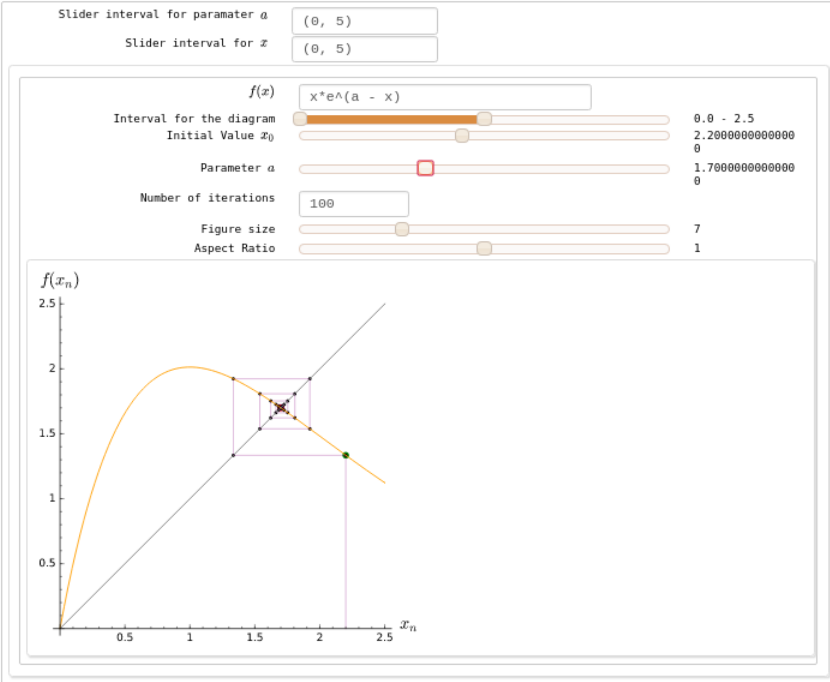


Figure 3: Cobweb Diagram [22] for $x_{n+1} = x_n e^{a-x_n}$ when $a = 1.7$ and $x_0 = 2.2$

Now, we present an interactive tool for generating cobweb diagrams. In Figure 3, a screenshot of the tool is given. By using this tool, the cobweb diagram of a difference equation $x_{n+1} = f(x_n; a)$ can be obtained for an controllable parameter a. First, intervals for the initial point x_0 and the parameter a are entered. After that, user inputs a function f into the corresponding input box. The interval of the diagram can be restricted by using a range slider. The user can also control the initial value x_0, the number of iterations, figure size, and aspect ratio of the graph. The green and red dots represent the initial and terminal points, respectively.

Depending on the given map and the initial point x_0, the diagram might be vertically or horizontally too long to see the graph properly. By changing the aspect ratio, one can obtain a better graph.

We can conclude from Figure 3 that the difference equation $x_{n+1} = x_n e^{a-x_n}$, when $a = 1.7$, has a positive fixed point and is locally asymptotically stable. Clearly, $x^* = a$ is a fixed point. One can obtain the same stability result by using the previous stability tool with $a = 1.7$.

Besides the cobweb diagram, time series are also an effective way to understand the dynamics of a discrete system. Figure 4 shows screenshots of the interactive tool representing the time series for the one-parameter difference equation $x_{n+1} = f(x_n; a)$. A choice of the interval for the parameter and the largest interval in which you want to see your initial value x_0 should be entered first. Then, after entering the function f, one can freely choose the initial value x_0 and the parameter a. The number of iterations, figure size, and aspect ratio can also be controlled.

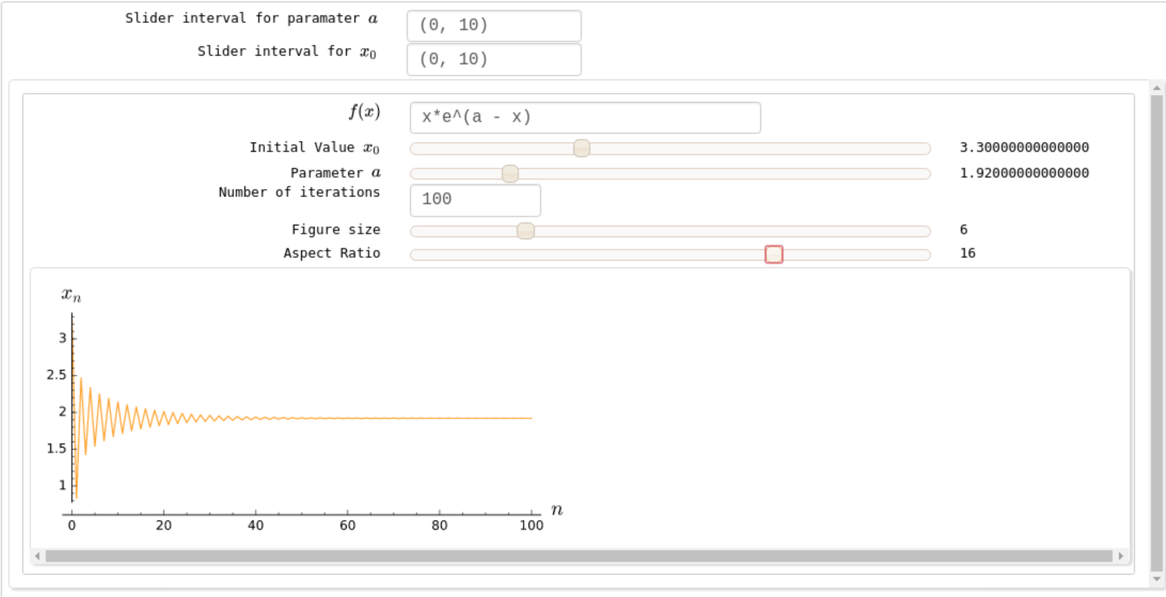


Figure 4: Time Series Diagram [23] for $x_{n+1} = x_n e^{a-x_n}$ when $a = 1.92$ and $x_0 = 3.3$

2.2 Two-Dimensional DDS

2.2.1 Phase Diagram

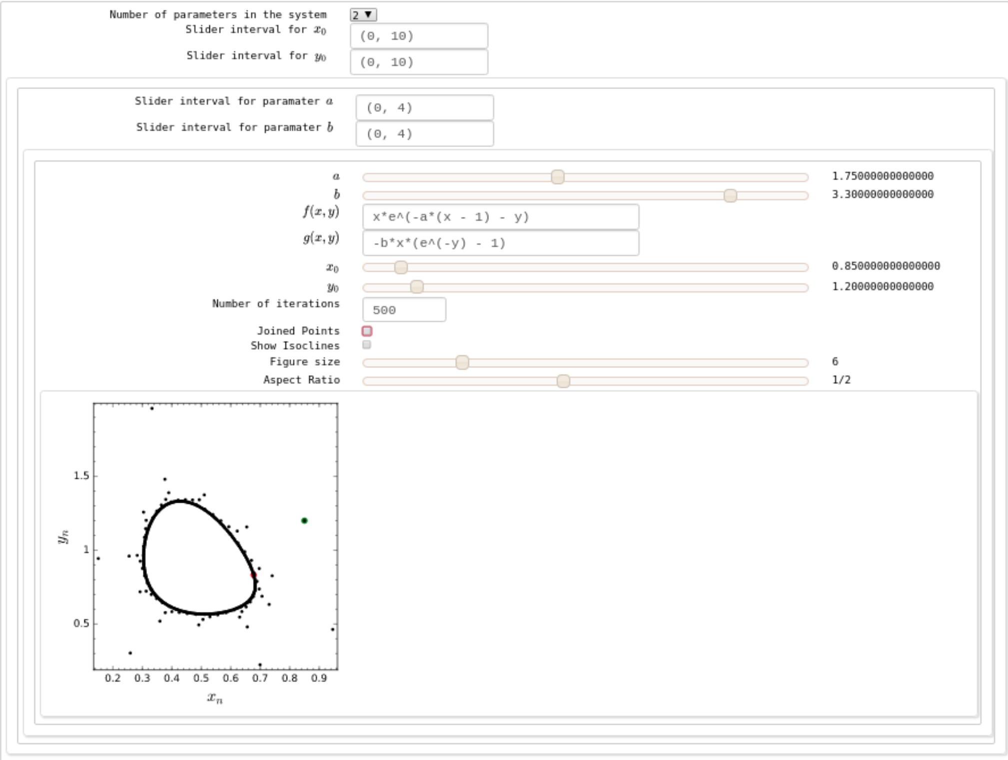
Phase (orbit) diagram (portrait/plane/space) is a graphical representation of the states of a dynamical system. In two-dimensional DDS, for each initial point, a phase diagram gives shows a set of ordered discrete points in the plane which is called an orbit of the system. Since they give important information about the stability and bifurcation, phase diagrams are highly important.

In this section, an interactive tool generating the phase (orbit) diagram (portrait/plane/space) of the following discrete-time system is introduced.

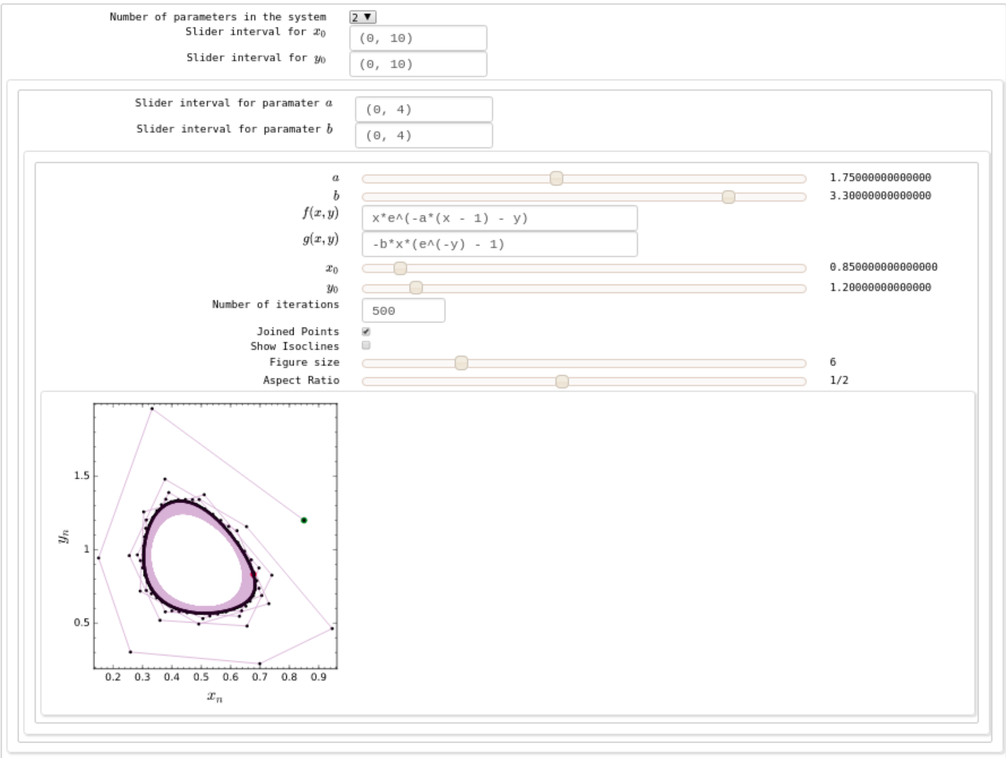
$$\begin{aligned} x_{n+1} &= f(x_n, y_n), \\ y_{n+1} &= g(x_n, y_n). \end{aligned} \tag{1}$$

Figure 5a and 5b represent screenshots of the tool. By using the pop-up menu on the top, up to four parameters can be added to the system to have richer dynamics and analyze the system qualitatively. The user enters the interval range for the parameters and the initial values x_0 and y_0 and then the functions $f(x, y)$ and $g(x, y)$. The number of iterations, figure size, and aspect ratio can also be controlled.

The discrete map given in the figures has two parameters, a and b. The diagrams show the orbit with initial point $(x_0, y_0) = (0.85, 1.2)$ after 500 iterations. In fact, Figure 5a and 5b show the same diagram except that the option *Joined Points* was not selected at Figure 5a. The user can also choose to see the isoclines of the map, namely the graphs of the curves $x = f(x, y)$ and $y = g(x, y)$, by checking the option *Show Isoclines*. The green and red points are the initial and the terminal points, respectively.



(a) Without Joined Points



(b) With Joined Points

Figure 5: Phase Diagram [24]

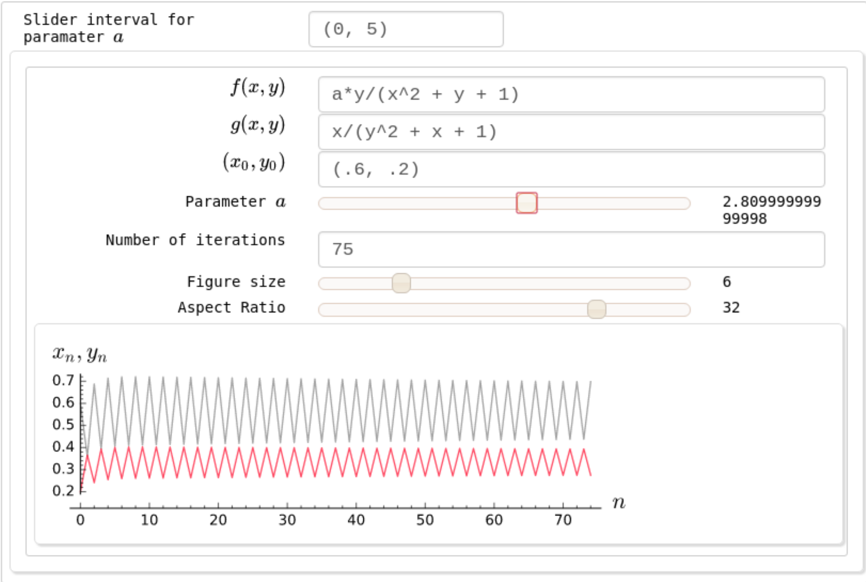


Figure 6: Time Series for 2D Maps [25]

2.2.2 Time Series

Another graphical representation of discrete systems is the time series diagram. We obtain stability information by using this diagram. The variables x_n and y_n can be analyzed separately as well.

Figure 6 shows a screenshot of the interactive tool presenting the time series of x_n and y_n. This tool enables the user to plot the time series for the dynamical system (1). First, the user enters an interval for the parameter and the intervals for x_0 and y_0. The number of iterations, figure size, and aspect ratio can be adjusted.

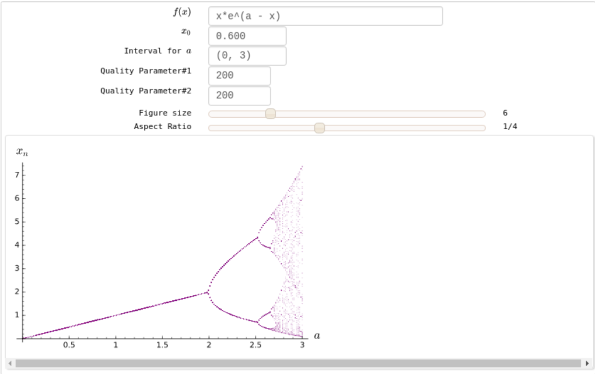
We observe from Figure 6 that, for the given parameter value, the system exhibits a period-2 orbit.

3 Bifurcation with SageMath

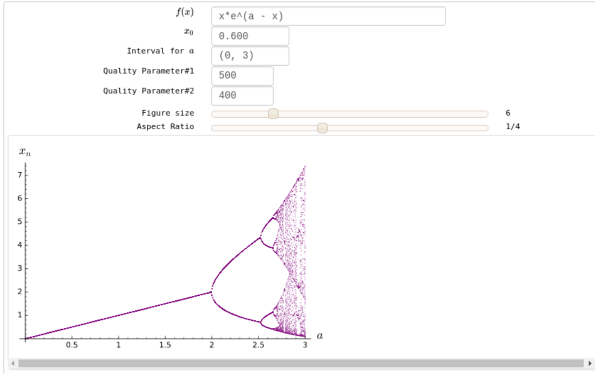
In order to see the big picture of a system, bifurcation diagrams are extremely informative. They give the impact of a particular parameter on the dynamics of the system. In this section, we introduce three interactive tools for bifurcation diagrams. The first diagram is for one-dimensional maps. Then, we introduce two tools for investigating the bifurcation diagrams of two-dimensional maps.

3.1 Bifurcation Diagram for One-dimensional Maps: Parameter-Variable Space

Figure 7 shows an interactive tool generating a bifurcation diagram of a given difference equation $x_{n+1} = f(x_n; a)$, for the parameter a. The user enters the initial condition x_0, the interval of the parameter to plot, and two parameters for increasing the quality of the plot. Quality Parameter 1 is



(a) Quality Parameters: 200-200



(b) Quality Parameters: 500-400

Figure 7: Bifurcation Diagram [26] for $x_{n+1} = f(x_n; a)$

the number of iterations performed, and Quality Parameter 2 is the number of points whose images are taken. Basically, the first and second parameters can be considered as the vertical and horizontal quality of the diagram, respectively. The figure size and the aspect ratio can also be controlled by the user. Since it usually takes time to obtain bifurcation diagrams, a low quality figure was chosen as a default diagram with parameters 200-200. For better quality figures, user can increase the quality parameters but should expect to wait for some time to obtain the diagram.

3.2 Bifurcation Diagram for Two-dimensional Maps: Parameter-Variable Space

Similar to the bifurcation diagrams for one-dimensional maps introduced in the previous section, for two-dimensional discrete systems, the tool for bifurcation gives very similar pictures, this time for two different variables x_n and y_n. In Figure 8, we observe that when $0 < a < 1$, the origin is asymptotically stable. For values of a between 1 and approximately 2.65 there is a stable fixed point, and after that value, there is a stable period-two orbit. Note that in the bifurcation diagram (Figure 8), we can observe the flip (period-doubling) bifurcation which agrees with the time series diagram in Figure 6.

3.3 Parameter Curve in Tr-Det Plane: Stability Interval and Bifurcation Types

In this section we introduce a novel geometric method for determining stability by using a parameter curve in the trace-determinant plane. The parameter curve is parametric: $\alpha(a) = (\text{tr}(a), \det(a))$, where trace and determinant are taken for the Jacobian matrix of the system at the fixed point. This tool is very informative when the fixed points are known explicitly. By using this tool, one can determine which parameters have what type of impact to the system.

In Figure 9, we investigate the stability of the fixed point $(0, 0)$ for the following one-parameter discrete system:

$$\begin{aligned} x_{n+1} &= a x_n + y_n, \\ y_{n+1} &= a^3 x_n + (a-1) y_n. \end{aligned} \tag{2}$$

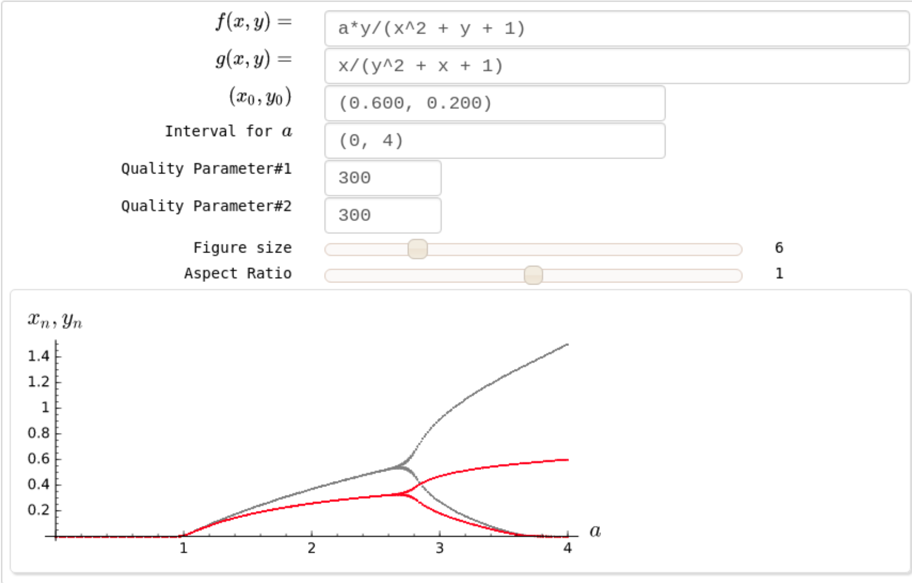
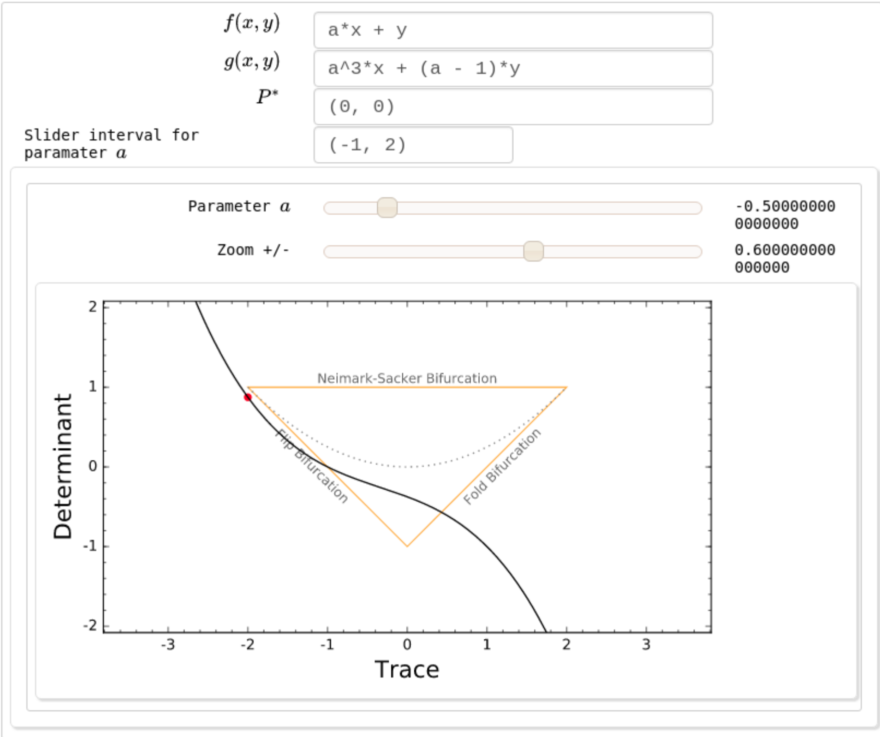


Figure 8: Bifurcation Diagram for 2D [27]

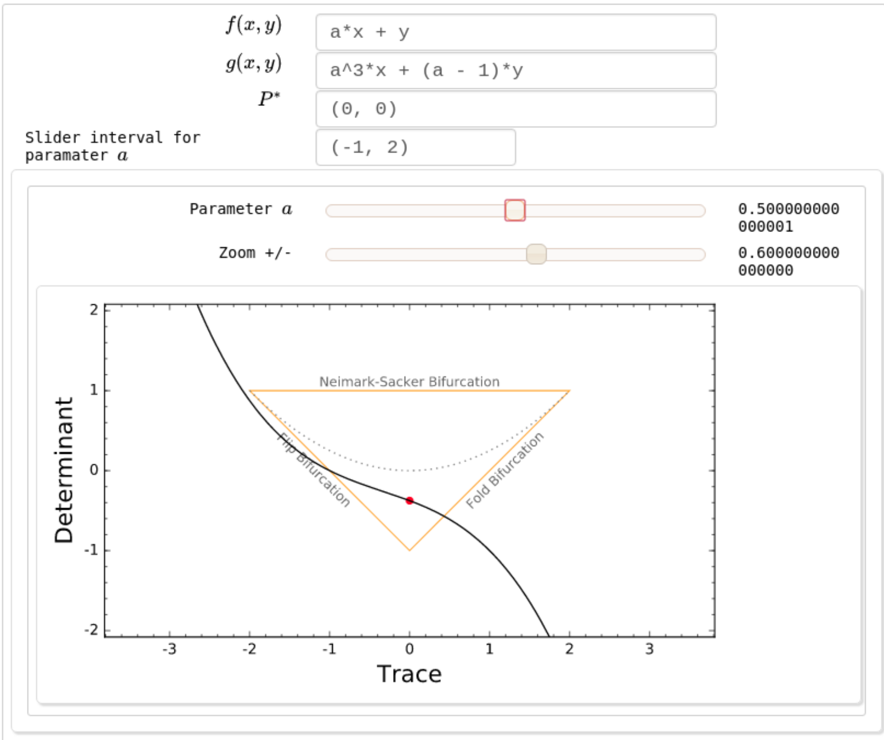
The interior region of the triangle is the stability region for the fixed point. The sides of the triangle are the bifurcation borderlines. The parameter curve gives all possible bifurcation types and stability values for parameter a. It can be seen in the figure that when $a = -0.5$, the fixed point $(0, 0)$ is unstable since the red point, which represents the trace-determinant point when $a = -0.5$, is outside the stability region. Since the parameter curve crosses the flip and fold bifurcation sides, parameter a causes two types of bifurcations: Fold and Flip Bifurcation. However, a does not cause the Neimark-Sacker Bifurcation.

It can also be determined that, when the fixed point is asymptotically stable, the eigenvalues for the parameter a are never complex, since the arc of the parameter curve inside the triangle is always below the dotted parabola. That is, no spiralling occurs for any values of the parameter a for the origin. Details on the Tr-Det plane can be found in [6].

The usual bifurcation diagram given in Figure 8 depends on the initial condition. Changing the initial value might give a totally different diagram. One of the strengths of the tool presented here is that it gives the local stability of the fixed point independent from the initial condition. However, the weak point of this bifurcation diagram is that it doesn't give any information on the periodic orbits or chaos. It focuses only on the bifurcations occurring at the borderline of stability and instability regions of a particular fixed point.



(a) $a = -0.5$



(b) $a = 0.5$

Figure 9: Tr-Det Plane and types of bifurcations caused by the parameter a [28]

4 Conclusion

Some mathematical and geometric interactive tools for the stability and bifurcation analysis of DDS were presented. For the stability of one-dimensional maps, not only hyperbolic but also non-hyperbolic fixed points were discussed. For planar maps, we presented a tool showing the phase diagram which can be controlled by up to 4 parameters. This is extremely important in order to understand the stability and bifurcation of dynamical systems. We also gave the time series and bifurcation diagram of one and two-dimensional DDS.

Among the interactive tools presented in this study, only the bifurcation diagram requires high computational speed. However, the default output of the bifurcation tools gives a sufficiently clear diagram. In order to obtain a higher-quality image, the user can increase the quality parameters, which will take more time to generate the diagrams.

All code was written in the open free computer algebra system *SageMath, Version 8.1*, running on a PC with Ubuntu Linux 17.10. The interactive tools and the codes are available online from the website `www.k-interact.net/dds` which was prepared by the author of this paper.

In the *Supplemental Electronic Materials* part of the references below, one can see the details about the tools and download `.txt` and `.sws` files by clicking the links. There are couple of ways to execute the codes. The `.sws` files can be executed in *Sage Notebook*. If SageMath is installed on your computer, you can load the file in *Sage Notebook* by *File > Load worksheet from a file*. The `.txt` files contain the codes. The simplest way to execute the codes is by downloading the `.txt` files and copying-pasting into `sagecell.sagemath.org`.

The presented tools are interactive, compact, accessible, and easy to use and understand. Furthermore, they don't require any programming skills. Although we haven't had a chance to conduct any pedagogical studies, we hope that these interactive tools will help students achieve a better understanding of discrete dynamical systems, difference equations, and their applications. Moreover, the researchers studying DDS in the area of mathematics, engineering, physics, biology, economics, will find the tools helpful.

As a further study, we will improve the tools and may add more technical options if necessary. We will also create a library for some well-known one- and two-dimensional discrete systems as well as various biological and business models. Furthermore, we will investigate the invariant manifolds, bifurcation diagrams in parameter-parameter plane, periodic orbits, and basin of attraction for DDS with SageMath.

References

[1] Allen, L., *An Introduction to Mathematical Biology*, Prentice Hall, 2007.

[2] Atehortúa, A. M., Ladino, L. M., and Valverde, J. C., *DSamala toolbox software for analysing and simulating discrete, continuous, stochastic dynamic systems.* Ingeniería e Investigación 32.2 (2012) 51-57.

[3] Dannan, F. M., Elaydi, S., and Ponomarenko, V., *Stability of hyperbolic and nonhyperbolic fixed points of one-dimensional maps.* Journal of Difference Equations and Applications 9.5 (2003): 449-457.

[4] Devaney, R.L., *An introduction to chaotic dynamical systems,* Vol. 13046. Reading: Addison-Wesley, 1989.

[5] Elaydi, S., *An Introduction to Difference Equations*, Springer, 2000.

[6] Elaydi, S., *Discrete Chaos: With Applications in Science and Engineering, Second Edition*, Chapman & Hall/CRC, 2008.

[7] Eröcal, B., and Stein, W., *The Sage Project: Unifying free mathematical software to create a viable alternative to Magma, Maple, Mathematica and MATLAB.* International Congress on Mathematical Software. Springer, Berlin, Heidelberg, 2010.

[8] Hutz, B., *Determination of all rational preperiodic points for morphisms of PN*, Mathematics of Computation, (2015) 84(291), 289-308.

[9] Kim, D-S, Markowsky, G., and Sang-Gu L., *Mobile Sage-Math for linear algebra and its application.* Electronic Journal of Mathematics and Technology 4.3 (2010): 285-298.

[10] Kulenovic,M.R., and Merino, O., *Discrete Dynamical Systems and Difference Equations with Mathematica*, Chapman & Hall/CRC, 2002.

[11] Kuznetsov, Y.A., *Elements of Applied Bifurcation Theory*, New York: Springer-Verlag. ISBN 0-387-94418-4, 1995.

[12] Lynch, S., *Dynamical Systems with Applications using MATLAB*, Birkhäuser Basel, 2004.

[13] Lynch, S., *Dynamical Systems with Applications using Maple*, Birkhäuser Basel, 2009.

[14] Morante, A., and Vallejo, J. A., *Chaotic dynamics with Maxima*, Electronic Journal of Mathematics & Technology, (2013) 7(4). Chicago

[15] Murakami, K., *Stability for non-hyperbolic fixed points of scalar difference equations*, Journal of mathematical analysis and applications, (2005) 310(2), pp.492-505.

[16] Scheinerman, E.R., *Invitation to dynamical systems,* Upper Saddle River, NJ: Prentice-Hall, 1996.

[17] Stein, W. *Sage: Creating a Viable Free Open Source Alternative to Magma, Maple, Mathematica, and MATLAB*, In F. Cucker, T. Krick, A. Pinkus, & A. Szanto (Eds.), Foundations of Computational Mathematics, Budapest 2011 (London Mathematical Society Lecture Note Series, pp. 230-238). Cambridge: Cambridge University Press. doi:10.1017/CBO9781139095402.011

[18] Ufuktepe, U., and Kapcak, S., *Applications of Discrete Dynamical Systems with Mathematica*, RIMS Kyoto Proceeding, (2014) 1909, 207-216.

[19] Ufuktepe, U., and Kulahcioglu, B., *Stability Analysis for Discrete Dynamical Models using Mathematica.* VFAST Transactions on Mathematics 6.2 (2015): 29-36.

Software Packages

[20] *SageMath, the Sage Mathematics Software System (Version 8.1)*, The Sage Developers, 2017, `http://www.sagemath.org`.

Supplemental Electronic Materials

[21] Kapçak, S., *SageMath file showing the stability of one-dimensional maps* (`.txt`) (Interactive)

[22] Kapçak, S., *SageMath file showing the cobweb diagram for one-dimensional maps* (`.txt`) (Interactive)

[23] Kapçak, S., *SageMath file showing the time series diagram for one-dimensional maps* (`.txt`) (Interactive)

[24] Kapçak, S., *SageMath file showing the phase diagram for two-dimensional maps* (`.txt`) (Interactive)

[25] Kapçak, S., *SageMath file showing the time series diagram for two-dimensional maps* (`.txt`) (Interactive)

[26] Kapçak, S., *SageMath file showing the bifurcation diagram for one-dimensional maps* (`.txt`) (Interactive)

[27] Kapçak, S., *SageMath file showing the bifurcation diagram for two-dimensional maps* (`.txt`) (Interactive)

[28] Kapçak, S., *SageMath file showing the parametric parameter curve in tr-det plane for two-dimensional maps* (`.txt`) (Interactive)

Appendix A Basic Concepts on DDS

In this appendix, we present the basic definitions and theorems of discrete-time systems. Most of the definitions and theorems given here are taken directly from [16, 6, 11, 1].

Consider the discrete dynamical system

$$\mathbf{x}_{n+1} = f(\mathbf{x}_n), \tag{3}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. A point $\mathbf{x}^*$ is said to be a *fixed point* of equation (3) if $f(\mathbf{x}^*) = \mathbf{x}^*$. There are three types of fixed points a system may possess. A fixed point $\mathbf{x}^*$ is called *asymptotically stable* if the following holds: For all starting values $\mathbf{x}_0$ near $\mathbf{x}^*$, the system not only stays near $\mathbf{x}^*$ but also $\mathbf{x}_n \to \mathbf{x}^*$ as $n \to \infty$. A fixed point $\mathbf{x}^*$ is called *stable* if for all starting values $\mathbf{x}_0$ near $\mathbf{x}^*$, the system stays near $\mathbf{x}^*$ but does not converge to $\mathbf{x}^*$. A fixed point $\mathbf{x}^*$is called *unstable* if it is neither asymptotically stable nor stable.

A.1 Stability for One-dimensional Maps

Let $x^* \in \mathbb{R}$ be a fixed point of the difference equation $x_{n+1} = f(x_n)$. A fixed point x^* of a map f is said to be *hyperbolic* if $|f'(x^*)| \neq 1$. Otherwise, it is *nonhyperbolic*.

Theorem 1 *Let x^* be a hyperbolic fixed point of a map f, where f is continuously differentiable at x^*. The following statements then hold true:*

1. *If $|f'(x^*)| < 1$, then x^* is asymptotically stable.*
2. *If $|f'(x^*)| > 1$, then x^* is unstable.*

Theorem 2 *Let x^* be a fixed point of a map f such that $f'(x^*) = 1$. If $f'(x)$, $f''(x)$, and $f'''(x)$ are continuous at x^*, then the following statements hold:*

1. *If $f''(x^*) \neq 0$, then x^* is unstable.*
2. *If $f''(x^*) = 0$ and $f'''(x^*) > 0$, then x^* is unstable.*
3. *If $f''(x^*) = 0$ and $f'''(x^*) < 0$, then x^* is asymptotically stable.*

Theorem 3 *Let x^* be a fixed point of a map f such that $f'(x^*) = -1$. Set $Sf(x^*) = -f'''(x^*) - \frac{3}{2}[f''(x^*)]^2$. If $f'(x)$, $f''(x)$, and $f'''(x)$ are continuous at x^*, then the following statements hold:*

1. *If $Sf(x^*) < 0$, then x^* is asymptotically stable.*
2. *If $Sf(x^*) > 0$, then x^* is unstable.*

For most of the one-dimensional discrete maps, the above theorems will be sufficient to determine the stability of fixed points. However, for example, they don't give any stability information when $f'''(x^*) = 0$ or $Sf(x^*) = 0$. In [3] and [15], authors present a solid theory of non-hyperbolic fixed points of continuous maps, and we have used those cases to create a complete tool for stability.

A.2 Stability for Two-Dimensional Maps

Theorem 4 *Consider the system of difference equations*

$$X_{n+1} = AX_n \tag{4}$$

where A is a 2×2 matrix. Denote $\rho(A) = \max\{|\lambda_1|, |\lambda_2|\}$ where λ_1 and λ_2 are the eigenvalues of the matrix A. The following statements hold for the equation (4).

(a) If $\rho(A) < 1$, then the origin is asymptotically stable.

(b) If $\rho(A) > 1$, then the origin is unstable.

Theorem 5 *In equation (4), the origin is asymptotically stable if the following condition holds true:*

$$|trA| - 1 < \det A < 1.$$

Theorem 6 *Let $f : G \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map, where G is an open subset of $\mathbb{R}^2$, X^* is a fixed point of f, and $A = Df(X^*)$. Then the following statements hold true:*

(a) If $\rho(A) < 1$, then X^* is asymptotically stable.

(b) If $\rho(A) > 1$, then X^* is unstable.

(c) If $\rho(A) = 1$, then X^* may or may not be stable.

A.3 Bifurcation

In this section, we present various types of changes in behaviour that may occur at bifurcation values.

The types of bifurcations depend on how the dynamics of a map change as a single parameter is varied. Consider a discrete-time dynamical system depending on a parameter

$$x \mapsto H(\mu, x), \quad x \in \mathbb{R}^n, \ \mu \in \mathbb{R},$$

where the map H is smooth with respect to both x and μ.

Let $x = x_0$ be a hyperbolic fixed point of the system for $\mu = \mu_0$. Let us monitor this fixed point and its eigenvalues of the Jacobian matrix of H evaluated at x_0 while this parameter varies. It is clear that there are, generically, only three ways in which the hyperbolicity condition can be violated. Either a simple positive eigenvalue approaches the unit circle and we have $\lambda_1 = 1$, or a simple negative eigenvalue approaches the unit circle and we have $\lambda_1 = -1$, or a pair of simple complex eigenvalues reach the unit circle and we have $\lambda_{1,2} = e^{\pm i\theta_0}$, $0 < \theta_0 < \pi$, for some value of parameter. Now, we give the following definitions.

Definition 7 *The bifurcation associated with the appearance of $\lambda_1 = 1$ is called a **fold** (or **tangent**) bifurcation.*

This bifurcation is also referred to as a **limit point**, **saddle-node bifurcation**, **turning point**, among others.

Definition 8 *The bifurcation associated with the appearance of* $\lambda_1 = -1$ *is called a* ***flip*** *(or* ***period doubling****) bifurcation.*

Definition 9 *The bifurcation corresponding to the presence of* $\lambda_{1,2} = e^{\pm i\theta_0}$, $0 < \theta_0 < \pi$, *is called a* ***Neimark-Sacker*** *(or* ***torus****) bifurcation.*

Notice that the fold and flip bifurcations are possible if $n \geq 1$, but for the Neimark-Sacker bifurcation we need $n \geq 2$.

Details on the types of bifurcations can be found in [6],[11].